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Hybrid mean value on some Smarandache functions

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Abstract: The mean value properties of the Smarandache function acting on k-th roots sequences is studied by using the elementary method, an interesting asymptotic formula is obtained.

Key words: smarandache function; k-th roots; mean value

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1 Introduction and conclusion

For any positive integer n, the Smarandache function Sdf(n) is defined as following:

$$Sdf(n) = \min\{m: m\in N, n \mid m!!\}.$$

Where $m!! = 2\cdot 4\cdot 6\cdot \cdots \cdot m$, if m is an even; $m!! = 1\cdot 3\cdot 5\cdot \cdots \cdot m$, if m is an odd. The other function $a_k(n)$ is denoted the integer part of k-th root of n. That is, $a_k(n) = [n^{1/k}]$, where $[x]$ is the greatest integer less than or equal to real number x.

These two function were both proposed by professor F. Smarandache in reference[1], where he asked us to study the properties of these function.

About the relations between the sequence and the Smarandache function, It seems that none had studied it, at least we have not seen any related papers before. However, about the properties of Sdf(n) and $a_k(n)$, many scholars showed great interest in reference[2-5].

In this paper, we study the hybrid mean value properties of the Smarandache function acting on the k-th roots sequences, and give an interesting asymptotic formula. That is, we shall prove the following conclusion.

Theorem 1 For any real number $x\geqslant 2$, we have the asymptotic formula:

$$\sum_{n\leqslant x} Sdf(a_k(n)) = \frac{7\pi^2}{12(k+1)}\frac{x^{(k+1)/k}}{\ln x} + O\left(\frac{x^{(k+1)/k}}{\ln^2 x}\right).$$

2 Some Lemmas

To complete the proof of the theorem, we need the following two simple Lemmas.

Lemma 1 If $2\nmid n$ and $n = p_1^{\alpha_1} p_2^{\alpha_2}\cdots p_k^{\alpha_k}$ is the factorization of n, where

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$p_1, p_2, \cdots, p_k$ are distinct odd primes and $\alpha_1, \alpha_2, \cdots, \alpha_k$ are positive integers, then

$$Sdf(n) = \max(S(p_1^{\alpha_1}), S(p_2^{\alpha_2}), \cdots, S(p_k^{\alpha_k})).$$

Proof Let $m_i = Sdf(p_i^{\alpha_i})$ for $i = 1, 2, \cdots, k$. Then we get $2 \nmid m_i$ ($i = 1, 2, \cdots, k$) and $p_i^{\alpha_i} \mid (m_i)!!$, $i = 1, 2, \cdots, k$. Let $m = \max(m_1, m_2, \cdots, m_k)$. Then we have $(m_i)!! \mid m!!$, $i = 1, 2, \cdots, k$. Thus we get $p_i^{\alpha_i} \mid m!!$, $i = 1, 2, \cdots, k$.

Notice that $p_1, p_2, \cdots, p_k$ are distinct odd primes. We have $\gcd(p_i^{\alpha_i}, p_j^{\alpha_j}) = 1$, $1 \leqslant i < j \leqslant k$. Therefore, we obtain $n \mid m!!$. It implies that $Sdf(n) \leqslant m$.

On the other hand, by the definition of m, if $Sdf(n) < m$, then there exists a prime power $p_j^{\alpha_j}$ $(1 \leqslant j \leqslant k)$ such that $p_j^{\alpha_j} \nmid Sdf(n)!!$. We get $n \nmid Sdf(n)!!$, a contradiction. Therefore, we obtain $Sdf(n) = m$. This proves Lemma 1.

Lemma 2 For positive integer n $(2 \nmid n)$, let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ is the prime powers factorization of n and $P(n) = \max_{1 \leqslant i \leqslant k}\{p_i\}$. if there exists $P(n)$ satisfied with $P(n) > \sqrt{n}$, then we have the identity $Sdf(n) = P(n)$.

Proof First we let $Sdf(n) = m$, then m is the smallest positive integer such that $n \mid m!!$. Now we will prove that $m = P(n)$. We assume $P(n) = p_0$. From the definition of $P(n)$ and lemma 1, we know that $Sdf(n) = \max(p_0, (2\alpha_i - 1)p_i)$. Therefore we get

(ⅰ) If $\alpha_i = 1$, then $Sdf(n) = p_0 \geqslant n^{1/2} \geqslant (2\alpha_i - 1)p_i$;

(ⅱ) If $\alpha_i \geqslant 2$, then $Sdf(n) = p_0 > 2\ln n\, n^{1/4} > (2\alpha_i - 1)p_i$.

Combining (ⅰ)~(ⅱ), we can easily obtain $Sdf(n) = P(n)$. This proves Lemma 2.

Lemma 3 Let $x \geqslant 1$ be any real number, we have the asymptotic formula

$$\sum_{n \leqslant x} S(n) = \frac{\pi^2}{12}\frac{x^2}{\ln x} + O\left(\frac{x^2}{\ln^2 x}\right).$$

Where $S(n) = \min\{m: m \in N, n \mid m!\}$.

Proof See reference [5].

Lemma 4 Let $x \geqslant 2$ be any real number, we have the asymptotic formula

$$\sum_{n \leqslant x} Sdf(n) = \frac{7\pi^2}{24}\frac{x^2}{\ln x} + O\left(\frac{x^2}{\ln^2 x}\right).$$

Proof It is clear that

$$\sum_{n \leqslant x} Sdf(n) = \sum_{u \leqslant (x-1)/2} Sdf(2u+1) + \sum_{u \leqslant x/2} Sdf(2u). \tag{1}$$

For the first part, we let the sets A and B as following:

$$A = \{2u+1 \mid 2u+1 \leqslant x, P(2u+1) \leqslant \sqrt{2u+1}\},$$

and

$$B = \{2u+1 \mid 2u+1 \leqslant x, P(2u+1) > \sqrt{2u+1}\}.$$

Using the Euler summation formula, we get

$$\sum_{2u+1 \in A} Sdf(2u+1) \ll \sum_{2u+1 \leqslant x} \sqrt{2u+1}\ln(2u+1) \ll x^{3/2}\ln x. \tag{2}$$

Similarly, from the Abel's identity[6] and Lemma 2, we also get

$$\sum_{2u+1 \in B} Sdf(2u+1) = \sum_{\substack{2u+1 \leqslant x \\ P(2u+1) > \sqrt{2u+1}}} P(2u+1) =$$

$$\sum_{1 \leqslant 2l+1 \leqslant \sqrt{x}} \sum_{2l+1 \leqslant p \leqslant x/(2l+1)} p + O\left(\sum_{1 \leqslant 2l+1 \leqslant \sqrt{x}} \sum_{\sqrt{2l+1} \leqslant p \leqslant x/(2l+1)}\right) =$$

$$\sum_{1 \leqslant 2l+1 \leqslant \sqrt{x}} \left[\frac{x}{2l+1}\pi\left(\frac{x}{2l+1}\right) - (2l+1)\pi(2l+1) - \int_{\sqrt{x}}^{x/(2l+1)} \pi(s)\,ds\right] + O(x^{3/2}\ln x), \tag{3}$$

where $\pi(x)$ denotes all the numbers of prime which is not exceeding x. Notice that $\pi(x) = x/\ln x + O(x/\ln^2 x)$

and

$$\sum_{1\le 2l+1\le\sqrt{x}}\left(\frac{x}{2l+1}\pi\left(\frac{x}{2l+1}\right)-(2l+1)\pi(2l+1)-\int_{\sqrt{x}}^{x/(2l+1)}\pi(s)\,ds\right)=$$
$$\sum_{1\le 2l+1\le\sqrt{x}}\left(\frac{1}{2}\frac{x^2}{(2l+1)^2\ln(x/(2l+1))}-\frac{1}{2}\frac{(2l+1)^2}{\ln(2l+1)}+O\left(\frac{x^2}{(2l+1)^2\ln^2(x/(2l+1))}\right)+\right.$$
$$\left.O\left(\frac{(2l+1)^2}{\ln^2(2l+1)}\right)+O\left(\frac{x^2}{(2l+1)^2\ln^2(x/(2l+1))}-\frac{(2l+1)^2}{\ln^2(2l+1)}\right)\right). \tag{4}$$

Hence

$$\sum_{1\le 2l+1\le\sqrt{x}}\frac{x^2}{(2l+1)^2\ln x/(2l+1)}=\sum_{1\le l\le(\sqrt{x}-1)/2}\frac{x^2}{(2l+1)^2\ln x/(2l+1)}=$$
$$\sum_{0\le l\le(\ln x-1)/2}\frac{x^2}{(2l+1)^2\ln x}+O\left(\sum_{\ln x-1/2\le l\le(\sqrt{x}-1)/2}\frac{x^2\ln(2l+1)}{(2l+1)^2\ln^2 x}\right)=$$
$$(\pi^2/8)(x^2/\ln x)+O(x^2/\ln^2 x). \tag{5}$$

Combining (2), (3), (4) and (5) we obtain

$$\sum_{u\le(x-1)/2}S(2u+1)=\frac{\pi^2}{8}\frac{x^2}{\ln x}+O\left(\frac{x^2}{\ln^2 x}\right). \tag{6}$$

For the second part, we notice that $2u=2^\alpha n_1$ where α, n_1 are positive integers with $2\nmid n_1$, let $S(2u)=\min\{m \mid 2u \mid m!\}$, from the definition of $Sdf(2u)$ and Lemma 3, we have

$$\sum_{2u\le x}Sdf(2u)=\sum_{\substack{2^\alpha n_1\le x\\ 2^\alpha>n_1}}Sdf(2^\alpha n_1)\ll\sum_{\alpha\le\ln x/\ln 2}\sqrt{x}\ll\sqrt{x}\ln x, \tag{7}$$

and

$$\sum_{2u\le x}Sdf(2u)=2\sum_{2u\le x}S(2u)+O(\sqrt{x}\ln x)=\frac{\pi^2}{6}\frac{x^2}{\ln x}+O\left(\frac{x^2}{\ln^2 x}\right). \tag{8}$$

Combining (7) and (8) we obtain

$$\sum_{u\le x/2}Sdf(2u)=\frac{\pi^2}{6}\frac{x^2}{\ln x}+O\left(\frac{x^2}{\ln^2 x}\right). \tag{9}$$

From (1), (6) and (9) we can get the result of Lemma 4.

3 Proof of the Theorem 1

For any real number $x\ge 1$, let M be a fixed positive integer such that $M^k\le x<(M+1)^k$, from the definition of $Sdf(n)$, we have

$$\sum_{n\le x}Sdf(a_k(n))=\sum_{t=1}^{M-1}\sum_{t^k\le n<(t+1)^k}Sdf(a_k(n))+\sum_{M^k\le n\le x}Sdf(a_k(n))=$$
$$\sum_{t=1}^{M-1}[(t+1)^k-t^k]\,Sdf(t)+\sum_{M^k\le n\le x}Sdf(M)=k\sum_{t=1}^{M}t^{k-1}Sdf(t)+O(x^{1/k}).$$

Let $B(y)=\sum_{n\le y}Sdf(n)$, by the Able's identity and Lemma 4, we can easily deduce that

$$\sum_{t=1}^{M}t^{k-1}Sdf(t)=M^{k-1}B(M)-(k-1)\int_2^M y^{k-2}B(y)\,dy=$$
$$M^{k-1}\cdot\frac{7\pi^2}{24}\frac{M^2}{\ln M}-(k-1)\int_2^M\frac{7\pi^2}{24}\frac{y^k}{\ln y}dy+O\left(\frac{M^{k+1}}{\ln^2 M}\right)=$$
$$\frac{7\pi^2}{24}\frac{M^{k+1}}{\ln M}-\frac{k-1}{k+1}\frac{7\pi^2}{24}\frac{M^{k+1}}{\ln M}+O\left(\frac{M^{k+1}}{\ln^2 M}\right)=$$
$$\frac{7\pi^2}{12(k+1)}\frac{M^{k+1}}{\ln M}+O\left(\frac{M^{k+1}}{\ln^2 M}\right).$$

Therefore, we can obtain the asymptotic formula:

$$\sum_{n\leqslant x} Sdf(a_k(n)) = \frac{7\pi^2}{12(k+1)}\frac{M^{k+1}}{\ln M} + O\left(\frac{M^{k+1}}{\ln^2 M}\right).$$

On the other hand, we also have the estimate

$$0 \leqslant x - M^k < (M+1)^k - M^k \ll x^{(k-1)/k}.$$

Now combining the above, we may immediately obtain the asymptotic formula

$$\sum_{n\leqslant x} Sdf(a_k(n)) = \frac{7\pi^2}{12(k+1)}\frac{x^{(k+1)/k}}{\ln x} + O\left(\frac{x^{(k+1)/k}}{\ln^2 x}\right).$$

This completes the proof of Theorem 1.

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Smarandache函数的混合均值

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摘要：用初等的方法研究了 Smarandache函数和 k次根序列的性质，并且得到了一个有趣的渐进公式.

关键词：Smarandache函数；k次根；均值

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